

QUADRATIC EQUATIONS IN GROUPS FROM THE GLOBAL GEOMETRY VIEWPOINT

ALEXANDER REZNIKOV

0. INTRODUCTION

Consider a group G and a system of equations

$$\begin{aligned}
 &w_1(x_1 \dots x_n) = z_1 \\
 &\dots \\
 (*) \quad &\vdots \\
 &w_k(x_1, \dots x_n) = z_k
 \end{aligned}$$

where w_1, \dots, w_k are some words in x_1, \dots, x_n . It is a classical problem to try to describe the set of solutions of (*) in G . If the system is quadratic in the sense of [14], then it is essentially equivalent to one quadratic equation $w(x_1, \dots, x_n) = z$, which can be transformed to one of the canonical forms

$$(**) \quad [x_1, x_2] \dots [x_{2g-1}, x_{2g}] = z$$

or

$$(***) \quad x_1^2 \dots x_g^2 = z.$$

Even for one homogeneous quadratic equation in a **free** group G , the question of describing all the solutions is nontrivial. It has been given much effort, starting with the work of Lyndon [17], see also Lyndon, Wicks [20], Zieschang [39], Lyndon, McDonough, Newman [19], Culler [6], Chmelevsky [4], Burns, Edmunds, Farvouqi [2], Commerford and Edmunds [5], Goldstein and Turner [10], Rosenberger [27], [28]. The survey of Schupp [31] describes the state of art by 1980. Recently, a major progress has been made, for G free, in describing

the set of solutions in the work of Piollet [22], Commerford and Edmunds [5], Grigorchuk, Kurchanov and Zieschang [10], and Olshansky [23]. The question concerning the solutions of a homogeneous quadratic equation in a free group, has been given a satisfactory answer in these papers. The technique involved uses the cancellation diagrams, described in details in [18], ch. V. Viewing a free group as a free product of infinite cyclic groups, it is quite natural to ask, whether it is possible to study the equations in a general free product, provided the information is given about the solutions in each of the factors. The final answer here is due to Rosenberger [27] with previous contribution of Commerford and Edmunds [5] and Burns, Edmunds and Formanek [3].

In the different direction, Marc Culler [6] has developed a beautiful approach to inhomogeneous equations in free groups. He defined a genus of an element $z \in G'$ similar to the genus norm in the terminology of Thurston [35] and proved several deep results, e.g., on the growth of genus (z^p) as $p \rightarrow \infty$. In the same paper he generalized the finite orbit principle of Chmelevsky [4] to arbitrary genus, and found nice applications for endomorphisms of free groups.

Another important contribution has been made by Edmunds [7], see also [8]. This paper contains a decomposition, up to a homotopy, of a continuous map between surfaces into a product of a pinch and a branched covering. This in fact may be transformed to a statement about quadratic equation in surface groups. It is also shown in [7], that any degree zero map between closed surfaces S, S' may be homotoped to a map from S to a one-skeleton of S' , which is essentially a homomorphism of $\pi_1(S)$ to a free group. Thus knowing such homomorphisms, one knows the (homotopy classes of) degree zero maps between surfaces.

We also mention the paper of Shapiro and Sonn [32], who have found a very simple approach to the Lyndon's rank inequality [18]. They computed the rank of the coupling form in the first cohomology of a one-relator group, and used the elementary (linear) symplectic geometry to study epimorphisms on a free group.

Finally, a very general picture of algorithmic aspects for equations in free groups has been given in Razborov [24] (and, with more detailed information, in his thesis).

The present paper suggests somewhat unexpected approach to equations in groups

via deeper parts of Riemannian geometry of manifolds. The central analytic tool that we will use is an existence theorem for minimal surfaces in Riemannian manifolds satisfying some uniformity conditions generalizing well-known theorem of Sacks-Uhlenbeck [29] and Schoen-Yau [30]. It uses classical genus reduction process as described by Ville [36]. In our previous paper [25] we have used this kind of argument for proving Thurston-Gromov finiteness theorem for (geometrically) hyperbolic groups.

The power of the approach presented here allows us to cover nearly all the results, described above, on quadratic equations in free groups, homogeneous and inhomogeneous and to give far-going generalizations to free products and subgroups of geometrically hyperbolic groups. There will be given two different proves of the Classification Theorem for solutions of homogeneous quadratic equations in a free group. One fits a more general framework of the Rosenberger's Classification Theorem for equations in free products, which we approach from our point of view in the section 4.

The other proof is based on the work of Schoen and Yau concerning the minimal surfaces in compact three-manifolds of positive scalar curvature; it will be presented in the section 3. It applies to a very restricted class of free products (including all free groups) but gives much more precise information on the structure of solutions; it turns out that all solutions are “elementary” in a sense explained below in 3.2.

We then state and prove the Classification Theorem for solutions of quadratic homogeneous equations in geometrically hyperbolic groups. These are fundamental groups of complete manifolds of pinched negative curvature, satisfying the uniformity and growth conditions of 5.1. This is a subclass in the class of Gromov word-hyperbolic groups.

Our existence theorem on minimal surfaces reduces, in the case when the ambient space is a surface, to a version of the well-known result of Edmunds on decomposition of maps between surfaces, given in a very precise and geometric form (our branched convergings are actually conformal maps).

Next we turn to the inhomogeneous equations. This corresponds to the solutions of Plateau problem in the minimal surfaces theory. Using the current (de Rham) norm in the loop space we recover the result of Culler on genus (z^p) in much more general context of subgroups of hyperbolic groups, as far as his generalizations of Chmelevsky' finite orbit

principle. To apply our technique to actually free groups, we need to realize these as commutators subgroups of excellent fiber knot groups and to use in part the Thurston hyperbolization theorem for knot manifolds [34].

This work owes its existence to the help of many people. It was started in the most warmful and stimulating atmosphere during my visit to Ruhr-Universität Bochum in 1991–1992. The warmest thanks are due to Heiner Zieschang, Martin Lustig, Frank Levin, Jürgen Jost, Shicheng Wang as well as to Ilya Rips and Marina Ville for valuable discussions.

1. PRELIMINARIES ON RIEMANNIAN SURFACES

1.1. Most of the essential results of this paper equally holds for nonorientable surfaces as well as for orientable. For the reader's convenience we will adopt the following strategy: complete proof will be provided for the orientable case with the precise specification of the changes necessary to cover the case of nonorientable surfaces. The formulation of the main classification results are more complicated for the nonorientable case, basically because of the more involved ierarchy of simple closed curves on a nonorientable surface, see [38].

We begin by the description of the genus reduction process of Douglas-Courant. The presentation given here follows that of [36], [26].

1.2. LEMMA. *Let Σ^g , $g \geq 2$ be an orientable surface and let X be a CW complex. Let $f : \Sigma^g \rightarrow X$ be a continuous map. Then one of the following two possibilities holds:*

- (i) $f_* : \pi_1(\Sigma^g) \rightarrow \pi_1(X)$ is essentially injective
- (ii) *There exists a homotopy decomposition*

$$\begin{array}{ccc} \Sigma^g & \xrightarrow{f} & X \\ \searrow & & \nearrow \\ & \Sigma^{g_1} \vee \Sigma^{g_2} & \end{array}$$

with $2 - g = (2 - g_1) + (2 - g_2) + 2$, and f_* decomposes as

$$\begin{array}{ccc} \hat{\pi}_1(\Sigma^{g_1})_* \hat{\pi}_1(\Sigma^{g_2}) = \pi_1(\Sigma^g) & \xrightarrow{f_*} & \pi_1(X) \\ \searrow & & \nearrow \\ & \pi_1(\Sigma^{g_1}) * \pi_1(\Sigma^{g_2}) & \end{array}$$

where $\hat{\pi}_1(\Sigma^{g_i})$ is the (free) fundamental group of Σ^{g_i} punctured at one point.

PROOF: Recall that f_* is not essentially injective, means that $\text{Ker } f_*$ contains a class of a simple closed curve γ . If this curve γ is separating, we may pinch it to a point as shown in Fig 1.3 (left). The resulted space is a wedge and (ii) follows. If γ is not separating then([38]) it is a meridian of a handle as in the Fig 1.3 (right). Let δ be the longitude of the handle and ε be the attaching circle. Then, up to conjugacy in $\pi_1(\Sigma^g)$, $\varepsilon = [\gamma, \delta]$ so $f_*\varepsilon = [1, f_*\delta] = 1$. Since ε is separating, we may proceed as before.

1.4. DEFINITION: Let X be a CW-complex. let $H : \pi_2(X) \rightarrow H_2(X)$ be the Hurewitz map, and let $\overline{\pi_2(X)} = \text{Im}H$. Let $z \in H_2(X)/\overline{\pi_2(X)}$. The Thurston genus norm of z , denoted $\|z\|_g$ is the minimal absolute value of the Euler characteristic of a singular surface $f : \Sigma^g \rightarrow X$, representing z . It is allowed for the orientable surface Σ^g to be disconnected.

1.5. LEMMA. Let X and z be as above. Suppose $f : \Sigma^g \rightarrow X$ is a continuous map such that $\chi(\Sigma^g) = \|z\|_g$. Then the restriction of f on every connected component of Σ^g is essentially injective.

PROOF: See [26], lemma 1.

The following standard fact will be frequently used in the paper, c.f.Culler [6].

1.6. LEMMA. Let X be a CW-complex and let $[\gamma] \in [\pi_1(X), \pi_1(X)]$. Then there exists a map f of a surface Σ^g with a removed disc $D^2 \subset \Sigma^g$ to X , such that $f|_{\partial D^2} = \gamma$.

PROOF: Let $[\gamma] = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_m b_m a_m^{-1} b_m^{-1}$ in $\pi_1(X)$. The fundamental group of $\Sigma^m \setminus D^2$ is free on generators $x_1, y_1, \dots, x_m, y_m$, and $[\partial D^2] = [x_1, y_1] \dots [x_m, y_m]$. Any map $f : \Sigma^m \setminus D^2$ such that $f_*(x_i) = a_i$ and $f_*(y_i) = b_i$ may be homotoped to yield $f|_{\partial D^2} = \gamma$.

Now we specify the changes in lemma 1.3., to yield the nonorientable case.

1.7. LEMMA. Let Σ^g be a nonorientable surface and let $f : \Sigma^g \rightarrow X$ be a continuous map. Then one of the following possibilities hold:

- (i) f_* is essentially injective
- (ii) f may be homotoped to a decomposition

$$\begin{array}{ccc} \Sigma^g & \xrightarrow{f} & X \\ \searrow & & \nearrow \\ & \Sigma^{g_1} \vee \Sigma^{g_2} & \end{array}$$

with $\chi(\Sigma^g) = \chi(\Sigma^{g_1}) + \chi(\Sigma^{g_2}) - 2$, and f_* decomposes as

$$\begin{array}{ccc} \pi_1(\Sigma^g) = \hat{\pi}_1(\Sigma^{g-1}) \underset{\mathbb{Z}}{*} \hat{\pi}_1(\Sigma^{g_2}) & \longrightarrow & \pi_1(X) \\ & \searrow & \uparrow \\ & & \pi_1(\Sigma^{g_1}) * \pi_1(\Sigma^{g_2}) \end{array}$$

where $\hat{\pi}_1(\Sigma^{g_i})$ is the fundamental group of Σ^{g_i} punctured at one point

- (iii) There exists a pair $(\Sigma^{g'}, f')$ consisting of an orientable surface $\Sigma^{g'}$, and a map $f' : \Sigma^{g'} \rightarrow X$ such that (Σ^g, f) is (up to homotopy) obtained from $(\Sigma^{g'}, f')$ by the following surgery: remove a small disc $D^2 \subset \Sigma^{g'}$ such that $f'|_{D^2} = \text{const}$, glue the opposite points of ∂D^2 together to obtain Σ^g , and define f by restriction.
- (iv) There exists a pair $(\Sigma^{g'}, f')$ consisting a surface $\Sigma^{g'}$ and a map $f' : \Sigma^{g'} \rightarrow X$ such that (Σ^g, f) is obtained from $(\Sigma^{g'}, f')$ by the following surgery: remove two discs $D_1, D_2 \subset \Sigma^{g'}$ such that $f'|_{D_i} = \text{const}$, glue the boundaries of D_1 and D_2 together to obtain Σ^g , and define f by restriction.

PROOF: Suppose f is not essentially injective and $[\gamma] \in \text{Ker } f_*$ is a simple closed curve. If γ is separating, then (ii) holds. If γ is not separating, then either 2γ is homotopic to a separating curve [38], (3.5.7) and we proceed as before with γ replaced by 2γ , or one of the possibilities, described in [38], (3.5.8)–(3.5.10) holds. This corresponds to the cases (iii) and (iv) of the lemma.

2. THE FUNDAMENTAL EXISTENCE THEOREM FOR MINIMAL SURFACES

2.1. For a Riemannian manifold, M , we adopt the **uniformity and the growth conditions** in the form: the injectivity radius $i_x(M) \rightarrow \infty$ as $x \in M$ escapes all compact subsets of M and for some $R > 0$ and for any $x \in M$ there exists a uniformly by x bi-Lipschitz diffeomorphism of the Euclidean ball $B(0, R)$ on a neighborhood of x . The Morrey regularity theory implies, as in [30], that any W_2^1 energy minimizing map of a compact surface to M will be smooth, hence harmonic.

Let Σ^g , $g \geq 1$, be an orientable closed surface, and let γ be a simple closed curve in Σ^g .

2.2. DEFINITION: An elementary pinch is a continuous map δ , defined as follows. If $g \geq 2$,

then we assume that γ is separating, and δ is the map

$$\delta : \Sigma^g \rightarrow \Sigma^{g_1} \vee \Sigma^{g_2}$$

of Σ^g on the wedge of two closed orientable surfaces, obtained by contracting γ to a point.

If $g = 1$, then

$$\delta : \Sigma^1 \rightarrow S^1$$

is defined by contracting γ , viewed as a meridian of the torus Σ^1 to a point followed by a retraction on a longitude.

2.3 DEFINITION: A pinch $\delta : \Sigma^g \rightarrow \Sigma^{g_1} \vee \cdots \vee \Sigma^{g_r} \vee S_{(1)}^1 \vee \cdots \vee S_{(s)}^1$ is defined inductively as composition of elementary pinches.

2.4. THEOREM. *Let Σ^g be a closed oriented surface, and let M be a Riemannian manifold which satisfies the uniformity conditions and the growth conditions. Suppose that M is either compact or of pinched non-positive curvature. Let $f : \Sigma^g \rightarrow M$ be a continuous map. Then there exists a pinch $\delta : \Sigma^g \rightarrow \Sigma^{g_1} \vee \cdots \vee \Sigma_r^g \vee S_{(r+1)}^1 \vee \cdots \vee S_{(r+s)}^1$, a collection of (branched) minimal immersions $\varphi_i : \Sigma^{g_i} \rightarrow M$, $1 \leq i \leq r$, and a collection of continuous maps $\hat{\varphi}_j : S_{(j)}^1 \rightarrow M$, $1 \leq j \leq r$, such that the diagram*

$$\begin{array}{ccc} \pi_1(\Sigma^g) & \xrightarrow{f_*} & \pi_1(M) \\ \searrow & & \uparrow \hat{\varphi}_{1*} * \hat{\varphi}_{2*} * \cdots * \hat{\varphi}_{(r+s)*} \\ & & \pi_1(\Sigma^{g_1}) * \cdots * \pi_1(\Sigma^{g_r}) * \mathbb{Z} * \cdots * \mathbb{Z} \end{array}$$

is commutative. Here $\hat{\varphi}_i$, $1 \leq i \leq r$ are some homomorphisms, conjugate to φ_i in $\pi_1(M)$. If $\pi_2(M) = 0$, then f is freely homotopic to $\vee \varphi_i$.

PROOF: We begin by applying the lemma 1.3. possibly several times, to find a decomposition 2.4 where all φ_i , $1 \leq i \leq r$ are essentially injective maps. We need to show that φ_i may be replaced by minimal immersions $\tilde{\varphi}_i$ which induce up to conjugation in $\pi_1(M)$ the same homomorphisms of the fundamental groups. The lines which follow actually mimic the argument of Sacks-Uhlenbeck [26] and Schoen-Yau [30]. Fix metrics h_i on Σ^{g_i} . Using the direct method of variational calculus, we find W_2^1 energy minimizing maps $\psi_i : \Sigma^{g_i} \rightarrow M$ inducing (up to conjugacy) the same map of fundamental groups, as φ_i .

By the regularity theory of Morrey, all ψ_i are smooth, hence harmonic. Next, we let the conformal classes $[h_i]$ of the metrics h_i vary inside the Teichmuller space T_{6g_i-6} . Let $[h_i^{(\alpha)}]$ be the sequence of conformal classes, such the *Energy* $(\psi_i^{(\alpha)})$ decreases to its infimum taken over T_{6g_i-g} . Because of the conformal invariance of the energy, we may assume $h_i^{(\alpha)}$ to be the (uniquely defined in their conformal classes) hyperbolic metrics.

Then by the Collar theorem of we conclude, as in [27], that the minimal length of a closed geodesic on $(\Sigma^{g_i}, h_i^{(\alpha)})$ is bounded away from zero. Observe that here we need the uniformity conditions on M , to assume that the lengths of essential curves of M are bounded away from zero, and the essential injectivity of $(\varphi_i)_*$. By the Mumford compactness criterion, the sequence $[h_i^{(\alpha)}]$ contains a subsequence, which we immediately relabel $[h_i^{(\alpha)}]$, whose image in the moduli space M_{6g-6} converges. Twisting by a diffeomorphism $\varphi_i^{(\alpha)} : \Sigma^{g_i} \rightarrow \Sigma^{g_i}$ we arrive to a converging sequence, again relabeled $[h_i^{(\alpha)}]$, already in the Teichmuller space itself. Now, the estimates of [29] show that $\psi_i^{(\alpha)}$ are Hölder equicontinuous. Moreover, $Im(\psi_i^{(\alpha)})$ cannot escape all compact sets in M , since otherwise $\psi_i^{(\alpha)}$ could be lifted to the universal cover of M , which is imposible by the maximum principle []. So $\psi_i^{(\alpha)}$ contains a C^0 -converging subsequence, whose limit is minimal by [29],[30].

If $\pi_2(M) = 0$ then f is freely homotopic to the wedge $\bigvee \varphi_i$. In particular, if M is a closed surface, we recover the following well-known results of Edmonds [7] (the Main Theorem)

2.5. THEOREM (EDMONDS). *Any continuous map f between surfaces Σ and M can be decomposed, up to a homotopy, as a pinch followed by a branched covering.*

Another consequence is a theorem of M. Ville [36]: any element in $H_2(M)/\overline{\pi_2(M)}$ can be represented by a minimal surface. Here $\overline{\pi_2(M)}$ is the image of $\pi_2(M)$ in $H_2(M)$ under the Hurewitz homomorphism. Combining this with the theorem of Sacks-Uhlenbeck, we get that any element in $H_2(M)$ is represented by a minimal surface. We refer to our paper [26] for various geometric applications of this result.

Observe that the definition of a pinch, given in [7], differs slightly from ours.

THE THREE-DIMENSIONAL SURGERY AND HOMOGENOUS QUADRATIC

EQUATIONS IN FREE PRODUCTS OF SPECIAL TYPE

3.1. In this section, we apply the Existence Theorem for minimal surfaces to the study of the solutions of the quadratic equation (*) is a free product of groups specified below in 3.3. The idea is to realize such a group as a fundamental group of a compact three-manifold of positive scalar curvature and then to apply the Existence Theorem 2.4. On the other hand, the result of Schoen-Yau [30] states that there are no stable minimal surfaces of genus more than zero in such three-manifold. This will mean that the pinch δ in 2.4 acts actually to a wedge of circles, which corresponds to **elementary** homomorphism of the surface group, in the sense of the following definition. Throughout this section Σ^g stands for a closed **orientable** surface.

3.2. DEFINITION: Let G be a group and let $\rho : \pi_1(\Sigma^g) \rightarrow G$ be a homomorphism. We will call ρ to be an **elementary homomorphism** if there exists a set of canonical generators $(u_i, v_i)_{i=1}^g$ of $\pi_1(\Sigma^g)$ such that $\rho(u_i) = 1$ (then $\rho(v_i)$ may be completely arbitrary elements in G).

3.3. THEOREM. Suppose $G = \bigstar_{i=1}^r G_i$, where G_i runs through the following list: \mathbb{Z}_m , \mathbb{Z}_m , $m \geq 2$, $\mathbb{Z}_m \times D_n^*$, $(m, n) = 1$, $\mathbb{Z}_m \times T_n^*$, $(m, 6) = 1$, $\mathbb{Z}_m \times S_4$, $(m, 6) = 1$, $\mathbb{Z}_m \times A_5$, $(m, 30) = 1$. Then any homomorphism $\rho : \pi_1(\Sigma) \rightarrow G$ is elementary.

*Equivalently, any set of solutions of (**) is obtained from an elementary solution ($x_i = 1, y_i$ arbitrary) by a repetition of the Dehn-Nielsen transformations.*

Here T_n stands for the group $\{x, y, z | x^{3^n} = y^4 = 1, y^2 = z^2, xyx^{-1} = y, xzx^{-1} = yz, yzy^{-1} = z^{-1}\}$. The theorem 3.3. contains, as a special case $G = F_r$, the main result of Zieschang [39], Piollet [22], and Grigorchuk-Kurchanov-Zieschang [10].

3.4. LEMMA. Let G be a group from the list of the theorem 3.3. Then G can be realized as a fundamental group of a three-manifold of positive scalar curvature.

PROOF: Observe that $\mathbb{Z} = \pi_1(S^2 \times S^1)$ and $S^2 \times S^1$ can be given a product metric of $(S^2, can) \times (S^1, can)$ with positive scalar curvature.

Next, all other groups G_i act freely on S^3 [37], hence S^3/G_i admits desired metric. To deal with free products, we recall a strong result of Gromov-Lawson and Schoen-Yau,

which says that positive scalar curvature manifolds admit surgery in codimension at least three. In particular, a connected sum of three-manifolds of positive scalar curvature again carries a metric of positive scalar curvature. This proves the lemma, since the fundamental group of a connected sum is a free product.

PROOF OF THE THEOREM 3.3: Let M be a three-manifold with fundamental group G , which exists according to the previous lemma. Let $f : \Sigma^g \rightarrow M$ be a map which induces the homomorphism φ . By the Theorem 2.4 we know that ρ decomposes as $*\varphi_{i_*} \circ \delta_*$, where φ_i , $1 \leq i \leq r$, are stable minimal maps of surfaces of genus ≥ 1 , to M . However, by a result of Schoen-Yau [30], theorem a three-manifold of positive scalar curvature carries no stable minimal surface. Hence $r = 0$ and ρ decomposes as a pinch $\vee \rightarrow \vee S_{(i)}^1$ followed by a map of $\vee S_{(i)}^{(1)}$ to M . This precisely means that the homomorphism ρ is elementary.

3.5. REMARK: If G is a fixed finite group, then, of course, the number of homomorphisms $\rho : \pi_1(\Sigma^g) \rightarrow G$ is finite for any fixed g . However as g grows, it becomes a very difficult problem to describe all these homomorphisms (it is much easier to find their number, [21]). We note that if $H_2(G, \mathbb{Z}) \neq 0$, then there are homomorphisms which are not elementary in the sense of 3.2. The converse to this statement is wrong.

3.6. REMARK: Recall that a solution (x_i, y_i) of $(**)$ in G is called free, if the corresponding homomorphism $f : \pi_1(\Sigma) \rightarrow G$ factors through a free group [18]. Any elementary solution is clearly free. The converse is also true and follows from 3.3.

So the statement of 3.3 may be reformulated in the following way: for G as in 3.3. all solutions of $(**)$ are free.

4. THE CLASSIFICATION THEOREM FOR SOLUTIONS OF HOMOGENEOUS

QUADRATIC EQUATIONS IN FREE PRODUCTS

4.1. Here we address the problem of an algebraic description for solutions of homogeneous quadratic equations $(**)$, $(***)$ in a general free products $G_1 * G_2$ in terms of solutions in each of G_i , $i = 1, 2$. See [] [] for some partial result for small genus. We will prove here the general results of Rosenberger [27] below, which resolves this problem completely. For simplicity we work with finitely presented groups G_i . The obvious in-

duction argument extends our result to an arbitrary number of groups. The idea of our approach is to realize G_i as a fundamental group of a compact 5-manifold M_i and then to study maps of surfaces Σ^g to $M_1 \# M_2$ using the transversality. We have a good reason to believe that this approach will prove very useful in studying similar problems in relations to amalgamated products, where the connected sums will be replaced by a surgery along a submanifold.

4.2. THEOREM. *Let ρ be a homomorphism from a surface group $\pi_1(\Sigma^g)$, Σ^g oriented, to G . There exists a collection of homomorphisms $\rho_i : \pi_1(\Sigma^{g_i}) \rightarrow G_1$, $1 \leq i \leq r_1$, $\rho_i : \pi_1(\Sigma^{g_i}) \rightarrow G_2$, $r_1 + 1 \leq i \leq r_1 + r_2$, and $\rho_i : \mathbb{Z} \rightarrow G$, $r_1 + r_2 + 1 \leq i \leq r_1 + r_2 + s$, a pinch $\delta : \Sigma^g \rightarrow \vee \Sigma^{g_i} \vee S_{(i)}^1$, and a collection of elements z_i , $1 \leq i \leq r_1 + r_2 + s$ in G , such that ρ decomposes as*

$$\begin{array}{ccc} \pi_1(\Sigma^g) & \xrightarrow{\rho} & G \\ \searrow \delta_* & & \nearrow *(z_i \rho_i z_i^{-1}) \\ & \pi_1(\Sigma^{g_1}) * \cdots * \pi_1(\Sigma^{g_{r_1+r_2}}) * \mathbb{Z} * \cdots * \mathbb{Z} & \end{array}$$

4.3. LEMMA. *Let G be a finitely presented group. There exists a compact 5-manifold, M , such the $\pi_1(M) = G$ and $\pi_2(M) = 0$.*

PROOF OF THE THEOREM 4.2: We begin by realizing G_i as a fundamental group of a compact 5-manifold M_i with $\pi_2(M_i) = 0$, as above. Let $M = M_1 \# M_2$, then $G = \pi_1(M)$. Given $\rho : \pi_1(\Sigma^g) \rightarrow G$, fix $f : \Sigma^g \rightarrow M$ with $f_* = \rho$, up to a conjugacy in G . We denote by S a separating 4-sphere in M . First we note that $\pi_2(M) = 0$. This actually will follow from the argument below. Assuming this, we use 1.3. to deform f to a composition of a pinch ρ followed by a wedge of maps $\varphi_i : \Sigma^{g_i} \rightarrow M$, $1 \leq i \leq r$ and $\varphi_i : S_{(i)}^1 \rightarrow M$, $r + 1 \leq i \leq r + s$, such that φ_i , $1 \leq i \leq r$ are essentially injective. We claim that any φ_i , $1 \leq i \leq r$ is freely homotopic to a map, whose image lies in one of M_i . Indeed, smooth φ_i and make it transversal to S . This is possible since S is of codimension one in M . Let $\mu(\varphi_i)$ be the number of connected components of $\varphi_i^{-1}(S)$.

We assume $\mu(\varphi_i)$ is minimal possible in the homotopy class of φ_i . If $\mu(\varphi_i) = 0$, then the image of φ_i lies in one of M_i , and we are done. So we may assume $\varphi_i^{-1}(S)$ be nonempty. Let $C \subseteq \varphi_i^{-1}(S)$ be a connected component of $\varphi_i^{-1}(S)$. If $[C] = 0$ in $\pi_1(\Sigma^{g_i})$

then C bounds a disc, D , and we may assume that no other components of $\varphi_i^{-1}(S)$ are in D , otherwise we will replace C by one of these components. Then φ_i maps D to one of M_i , say M_1 , and we can push $\varphi_i|_D$ out of M_1 (here we need $\pi_2(M_i) = 0$), reducing $\mu(\varphi_i)$. This argument also shows that $\pi_2(M) = 0$. So C is an essential closed curve, and φ_i is not essentially injective, a contradiction. So φ_i may be homotoped to a map, whose image is in one of M_i . Back to the homomorphisms, this gives precisely the decomposition of 4.2. Q.E.D.

As an immediate corollary, we get another proof to the Classification Theorem for equations in free groups, c.f. 3.3.

4.5. We now turn to the study of homomorphisms of a nonorientable surface groups to $G_1 * G_2$, which corresponds to the equation (**). The crucial point in the proof of the Classification Theorem 4.2. below is contained in the following lemma.

LEMMA. *Suppose $f : \Sigma^g \rightarrow M_1 \# M_2$, is a map of an nonorientable surface of the genus g . Suppose f is not freely homotopic to a map, whose image is contained in one of M_i .*

Then $\text{Ker } f_$ contains a class of an essential closed separating curve.*

PROOF: First we deform f to a smooth map which is transversal to S , and relabel the new map by f again. The preimage $f^{-1}(S)$ is nonempty and is a disjoint union of simple closed two-side curves, say C_1, \dots, C_m . As in the proof of 4.2., we may think of all these curves to be essential. The homology class $[C_1] + \dots + [C_m]$ is zero in $H_1(\Sigma, \mathbb{Z}_2)$ hence if $m = 1$, then the (unique) curve C_1 is separating, and we are done. So we assume that $m \geq 2$ and non of C_i is separating. Cut Σ along C_1 . There are two possible cases:

- (i) The resulting surface is nonorientable. Then by [38], C_1 has representation $V_1 V_2$ in $\pi_1(\Sigma^g)$, where V_1, \dots, V_g is a set of generators for $\pi_1(\Sigma^g)$ with the canonical relation $V_1^2 \dots V_g^2 = 1$.

Since $f(C_i)$ is contained in the sphere S , we have obviously $f_*(C_i) = 1$ in G , so $f(V_1)f(V_2) = 1$, hence $f(V_1^2 V_2^2) = 1$. The element $V_1^2 V_2^2$ is represented by a closed separating curve in Σ^g and is contained in $\text{Ker } f_*$, as stated.

- (ii) The resulting surface $\tilde{\Sigma}$ is orientable. Then Σ^g is obtained by a surgery, described in 1.7(iv). Consider C_2 as a curve in $\tilde{\Sigma}$. If it is not separating, then it is a meridian of

a handle, and we immediately get a closed separating curve (and attaching circle of the handle) representing an element of $\text{Ker } f_*$, as in 1.3. So we may assume C_2 is separating (Fig 4.5). In the case of Fig 4.5, right, the curve C_2 is separating in Σ . So it is enough to study the case of Fig 4.5, left.

The surface Σ^g is therefore a Klein bottle with several handles and Möbious bands attached. We may localize these handles and Möbious bands within a disc D^2 in the Klein bottle. We claim the boundary circle ∂D^2 is a curve we need. Indeed, if is a closed separating essential curve, and we need only to prove that $[\partial D^2] \in \text{Ker } f_*$. Let C_3 be the longitude of the Klein bottle, then $[\partial D^2] = [C_3][C_2]C_3^{-1}[C_2]$, and since $f_*[C_2] = 1$, we get $f_*[\partial D^2] = 1$. Q.E.D.

4.6. We may now switch on the decomposition procedure of 1.7 using the previous lemma, and to produce the canonical form of the homomorphism f_* is the nonorientable case.

4.7. DEFINITION: Let Σ be a surface (possibly nonorientable). An elementary pinch is either a map, defined by 2.2, if Σ is not a Klein bottle, or a projective plane, or the (homotopically unique) map $f : \Sigma^2 \rightarrow \Sigma^1$ of the Klein bottle to S^1 , which induces the generator of $H^1(\Sigma^2, \mathbb{Z}) = \mathbb{Z}$, or the map $\mathbb{R}P^2 \rightarrow pt$, or the map $\Sigma^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \xrightarrow{id \vee \text{const}} \mathbb{R}P^2$.

A pinch $f : \Sigma \rightarrow \Sigma^1 \vee \dots \vee \Sigma^k \vee S^1 \dots \vee S^1$ is a composition of elementary pinches.

4.8. Now we state the Classification Theorem for homomorphisms $\rho : \pi_1(\Sigma) \rightarrow G_1 * G_2$.

THEOREM. Any homomorphism $\rho : \pi_1(\Sigma) \rightarrow G_1 * G_2$ admits a decomposition

$$\begin{array}{ccc} \pi_1(\Sigma) & \longrightarrow & G_1 * G_2 \\ \searrow \delta_* & & \nearrow *(z_i p_i z_i^{-1}) \\ \pi_1(\Sigma^{g_1}) * \dots * \pi_1(\Sigma^{g_{r_1+r_2}}) * \mathbb{Z} * \dots * \mathbb{Z} & & \end{array}$$

where δ is a pinch in the sense of 4.7., and ρ_i, z_i are as in 4.2.

PROOF: The only case which is not yet covered, concerns maps of the Klein bottle Σ^2 to M which are not essentially injective. Suppose $f : \Sigma^2 \rightarrow M$ is such a map and γ is an essential simple closed curve such that $[\gamma] \in \text{Ker } f_*$. Then by the classification of [38], we have the following cases.

- (i) $[\gamma] = V_1$. Then homotopically f factors through the pinch $\Sigma^2 \rightarrow \mathbb{R}P^2$.
- (ii) $[\gamma] = V_1 V_2$. Then homotopically f factors through the pinch $\Sigma^2 \rightarrow S^1$.

(iii) $[\gamma] = V_1^2$. Then homotopically f factors through the pinch $\Sigma^2 \rightarrow \mathbb{R}P^2 \vee \mathbb{R}P^2$.

5. CANONICAL STRATIFICATIONS IN MODULI SPACES OF SOLUTIONS

TO HOMOGENEOUS QUADRATIC EQUATIONS IN

GEOMETRICALLY HYPERBOLIC GROUPS

5.1. Throughout this section G is a fundamental group of a Riemannian manifold M with a sectional curvature $K(M)$ pinched as $-K \leq K(M) \leq -k < 0$ and satisfying the uniformity conditions and the growth conditions (e.g., compact). Such groups will be called geometrically hyperbolic. Observe that a f.g. free group is not a fundamental group of a compact manifold of negative curvature, but is a fundamental group of a manifold, satisfying the conditions above, e.g., an infinite cyclic covering of an excellent knot complement, see 5.6.

It turns out that the Classification Theorem for quadratic equations in free groups in a form proved in 3.3 and 4.2 is “almost true” for geometrically hyperbolic groups. To make it precise, we need a following definition.

5.2. DEFINITION: Let $\rho : \pi_1(\Sigma^g) \rightarrow G$, Σ^g oriented, be a homomorphism. We say that ρ has a defect at least d , denoted $def(\rho) \geq d$, if ρ admits a decomposition 2.4 with $s \geq d$. A maximal d such that $def(\rho) \geq d$ is called the defect of ρ .

5.3. THEOREM. *Let G be a geometrically hyperbolic group. Consider all homomorphisms $\rho : \pi_1(\Sigma^g) \rightarrow G$, of defect d , so that ρ admits a decomposition*

$$\begin{array}{ccc} \pi_1(\Sigma^g) & \longrightarrow & G \\ \delta_* \searrow & & \uparrow \varphi_1 * \cdots * \varphi_{r+d} \\ & & \pi_1(\Sigma^{g_1}) * \cdots * \pi_1(\Sigma^{g_r}) * \mathbb{Z} * \cdots * \mathbb{Z} \end{array}$$

with δ a pinch and φ_i , $1 \leq i \leq r$, essentially injective. Then for a fixed δ_ , there are only finite number of possibilities up to an automorphism of $\pi_1(\Sigma^{g_i})$ and conjugation in G for φ_i , $1 \leq i \leq r$.*

PROOF: Granted 2.4, it is enough to show that for a fixed g , there are only finite number of conjugate classes of essentially injective homomorphisms $\varphi : \pi_1(\Sigma^g) \rightarrow G$. This is a generalization of the well-known theorem of Thurston [35] and Gromov [13], where the actual

injectivity was demanded. In our previous paper [25], we gave an analytic proof, which in fact applies here without any changes. Indeed, consider a sequence of nonconjugate homomorphisms $\varphi_i : \pi_1(\Sigma^g) \rightarrow \pi_1(M) = G$, and let $f_i : \Sigma^g \rightarrow M$ be maps which induces φ_i up to conjugacy. Fix a metric h of curvature -1 on Σ^g and consider the harmonic maps $\bar{f}_i : \Sigma^g \rightarrow M$, inducing φ_i . This is possible by Sacks-Uhlenbeck [29] and Schoen-Yau [30]. Let $E_i^h(\bar{f}_i)$ be the energy of \bar{f}_i . By the Thurston-type inequality of [25], $Area(\bar{f}_i)$ stays bounded as $i \rightarrow \infty$, so we may perturb h to some h_i , again hyperbolic, such that $E_i^{h_i}(\bar{f}_i)$ is bounded. Now the argument of [30] shows that the class of h_i in the moduli space M_{6g-g} stays in some compact set (here we need the uniform conditions for M), so we can replace h_i by some fixed \bar{h} , twisting \bar{f}_i if necessary by a diffeomorphism of Σ^g such that $E_i^{\bar{h}}(\bar{f}_i)$ remains bounded. Applying the collar argument of [30] again, we see that for a given curve C in Σ^g , the minimal length of curves in the homotopy class of $\bar{f}_i(C)$ stays bounded. Since M satisfies growth conditions, there are but finite number of conjugacy classes in G which may be images of $[C]$ under f_{i*} . Since $\pi_1(\Sigma^g)$ is not a free product [18], there are but finite number of nonconjugates among φ_i . Q.E.D.

5.4. We move now to the description of homomorphisms of a nonorientable surface group to a geometrically hyperbolic group G . The key result which will be used in the Classification Theorem below, is contained in the following lemma.

LEMMA. *Suppose G is a geometrically hyperbolic group and Σ^g is a nonorientable surface, $g > 2$. Then there are, up to an automorphism of $\pi_1(\Sigma^g)$ and a conjugation of G , only finitely many homomorphisms $\varphi : \pi_1(\Sigma^g) \rightarrow G$, such that $\text{Ker } (\varphi)$ does not contain a class of a simple closed essential separating curve.*

PROOF: Let M be as above with $\pi_1(M) = G$. Fix an orientable double covering $\pi : \bar{\Sigma} \rightarrow \Sigma$. Consider a sequence of maps $f_i : \Sigma \rightarrow M$, which satisfy the condition of the lemma. Consider the maps $f_i \circ \pi : \bar{\Sigma} \rightarrow M$. We claim that even though $f_i \circ \pi$ may not be essentially injective, the proof of 5.3 goes through. Indeed, to apply the Mumford criterion we need to show that no simple closed geodesic \bar{C}_i of $(\bar{\Sigma}, h_i)$ may have a decreasing to zero length. Observe first that the metrics h_i may be chosen equivariant under the canonical involution σ of $\bar{\Sigma}$, so that $h_i = \pi * \tilde{h}_i$, where \tilde{h}_i is a metric on Σ . Suppose $length(\bar{C}_i) \rightarrow 0$.

The image $C_i = \pi(\bar{C}_i)$ is a closed geodesic and $length_{\tilde{h}_i}(C_i) \rightarrow 0$. Hence there exists a **simple** closed geodesic, which we relabel C_i , such that $length_{\tilde{h}_i}(C_i) \rightarrow 0$. Since the metric is hyperbolic, it is automatically essential. Now, either C_i or C_i^2 lifts to a simple closed geodesic of $\bar{\Sigma}$ and the argument of [30] shows that $[f_i(C_i^2)] = 0$ for i big enough in G . Since G is torsion-free, $C_i \in \text{Ker } f_{i*}$. We will show later, that this is impossible. So the Mumford criterion applies, and we may twist f_i by a diffeomorphism say g_i of $\bar{\Sigma}$, which by the argument at the end of 6.6 can be taken σ -equivariant. and get a metric, relabeled h_i , which stays in a compact subset of the Teichmuller space. Then again as in 5.3, we find a σ -invariant metric h of $\bar{\Sigma}$, such that the energy of f_i is bounded. The collar argument of [30] shows that for any simple closed geodesic of $\bar{\Sigma}$ the minimal length in the homotopy class of its image under f_i stays bounded. This is still true for any simple closed geodesic C in Σ , since either C or C^2 lifts to a simple curve in $\bar{\Sigma}$. The proof is now completed as in 5.3.

So we need only to show that f_i is essentially injective, i.e., $C_i \in \text{Ker } f_{i*}$ is impossible. Relabel f_i by f and C_i by C . Suppose $C \in \text{Ker } f_*$. By the classification of simple closed curves of [38] we encounter the following possibilities (we use the terminology of [38]):

- (i) $[C] = V_1$. Then V_1^2 is represented by a simple separating loop and $V_1^2 \in \text{Ker } f_*$, contrary to the hypothesis of the lemma.
- (ii) $[C] = V_1 \dots V_g$ (g even). This means that after cutting Σ along C we get an orientable surface $\hat{\Sigma}$. Take a separating curve D in $\hat{\Sigma}$ as in Fig 4.5, left, then as we have seen, $DCD^{-1}C$ is represented by a simple separating loop and $DCD^{-1}C \in \text{Ker } f_*$.
- (iii) $[C] = V_1 V_2$. Then $V_1^2 V_2^2$ is represented by a simple separating loop and $V_1^2 V_2^2 \in \text{Ker } f_*$.
- (iv) $[C] = V_1 \dots V_g$ (g odd). Then $[C^2]$ is represented by a simple separating loop and $[C^2] \in \text{Ker } f_*$.
- (v) C is separating, which contradicts the hypothesis. Q.E.D.

5.5. We are now ready to state the Classification Theorem for homomorphisms of a nonorientable surface group to G .

THEOREM. *Let G be a geometrically hyperbolic group, let Σ^g be a nonorientable surface*

and let $\rho : \pi_1(\Sigma^g) \rightarrow G$ be a homomorphism. There exists a decomposition

$$\begin{array}{ccc} \pi_1(\Sigma^g) & \longrightarrow & G \\ \searrow \delta_* & & \uparrow \varphi_1 * \cdots * \varphi_{r+s} * 1 * \cdots * 1 \\ \underbrace{\pi_1(\Sigma^{g_1}) * \cdots * \pi_1(\Sigma^{g_r})}_r * \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_s * \underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_q \end{array}$$

where $\Sigma^{g_1} \cdots \Sigma^{g_r}$ are not projective planes, δ is a pinch and $\varphi_1, \dots, \varphi_r$ are essentially injective. There are, up to automorphisms of Σ^{g_i} and conjugations in G , only finitely many possibilities for homomorphisms φ_i , $1 \leq i \leq r$.

5.6. EXAMPLE: Let $K \subset S^3$ be an excellent knot. The knot manifold $S^3 \setminus K$ admits a hyperbolic structure of a finite volume, which does not satisfy 5.1. Consider the universal cyclic covering $V \rightarrow S^3 \setminus K$, so that $\pi_1(V) = G$, where G is the knot group of K . This is a hyperbolic manifold which has just one end, which is a cylinder $S^1 \times \mathbb{R}$. It is easy to check that V satisfies the uniformity and the growth conditions, so 5.3 and 5.5 applies for G . Recall that G is either f.g. and free or an infinite amalgam.

6. THE DIRICHLET PROBLEM, THE CURRENT NORM AND

THE GENUS ESTIMATES IN

GEOMETRICALLY HYPERBOLIC GROUPS

6.1. In his paper [6], M. Culler proved the following remarkable results for commutators in free group F_r .

6.1.1. THEOREM (CULLER). *Let $w \in F'_r$ and let $\text{genus}(w)$ be the minimal number of simple commutators, whose product equals w . Then $\text{genus}(w^p)$ grows linearly with p .*

6.1.2. THEOREM (CULLER). *Let $w \in F'_r$ and let $\text{genus}(w) = g$. There are only finite number of equivalence classes of solutions to the equation $[x_1, y_1] \cdots [x_g, y_g] = w$ under the action of $C_w \times A_w$, where the centralizer C_w acts by conjugation, and A_w is the subgroup of $\text{Aut}(F_{2g})$, fixing the word $r = [x_1, y_1] \cdots [x_g, y_g]$.*

The group A_w is an extension of the mapping class group \mathcal{M}_g [18].

In this section, we will prove theorem 6.2 and below, which extend 6.1.1. and 6.1.2. to all geometrically hyperbolic groups.

6.2. THEOREM. *Let G be geometrically hyperbolic and let $w \in G'$. Then $\text{genus}(w^g)$ grows linearly with p .*

6.3. LEMMA. *Let M be a Riemannian manifold satisfying 5.1. Then any conjugate class in $\pi_1(M)$ is realized by a closed geodesic.*

PROOF: This is well known (see [9], for example).

6.4. THEOREM (DOUGLAS-RADO-MORREY-LEMAIRE-JOST). *Let M be a Riemannian manifold satisfying 5.1. and let γ be an smooth closed curve in M . There exists a harmonic map of $\Sigma^g \setminus D$ in M such that $\partial(\Sigma^g \setminus D)$ is mapped monotonically in γ . Here $g = \text{genus}([\gamma])$ and the metric of $\Sigma^g \setminus D$ is chosen arbitrarily.*

PROOF: See [15],[16].

6.5. DEFINITION: Let γ be a closed curve in M . Denote $||\gamma||_\infty^{-1}$ to be the current norm $\sup_{\omega \in \Omega^2(M)} \frac{\int_\gamma \omega}{||\omega||_\infty^1}$, where $||\omega||_\infty^1 = \sup_M |d\omega|$.

PROOF OF THE THEOREM 6.2.: First we choose a closed geodesic γ with $[\gamma] = \omega$ which exists by 6.3. Let $\Sigma^g \setminus D \hookrightarrow M$ be a harmonic map spanning γ , which exists by 6.4. By the Gauss formula we have $K(\Sigma^g \setminus D) \leq -k$ in all regular points. Let $\omega \in \Omega^2(M)$. Applying the Stocks formula, we get

$$\int_\gamma \omega = \int_{\Sigma^g \setminus D} d\omega \leq ||d\omega||_\infty \text{Area}(\Sigma^g \setminus D).$$

The Sharp Thurston Inequality of [25] applies to $\Sigma^g \setminus D$ to give

$$\text{Area}(\Sigma^g \setminus D) \leq \frac{2\pi}{k}(2g - 1)$$

So

$$(\text{****}) \quad 2g - 1 \geq \frac{k}{2\pi} ||\gamma||_\infty^{-1}.$$

Applying (****) to γ , iterated p times, we get

$$2\text{genus}(\omega^p) - 1 \geq \frac{k \cdot p}{2\pi} ||\gamma||_\infty^{-1},$$

as desired. Observe that yet $p \cdot \gamma$, and maybe γ itself, is not a Jordan curve, we may slightly perturb it to produce a Jordan curve with arbitrarily small integral curvature, which is a boundary term in (****) so that the argument goes through.

PROOF OF THE THEOREM 6.1.1: Take M to be the universal cyclic covering of the knot manifold of an excellent fiber knot, so that G' is finitely generated and hence free [1]. Then M admits a hyperbolic structure as in 5.5. and 6.2. applies directly.

6.6. THEOREM. *Let G be geometrically hyperbolic and let $w \in G'$ be of genus g . Then there are only finite number of equivalence classes of solutions to the equation $[x_1, y_1] \cdots [x_g, y_g] = w$ under the action of \mathcal{M}_g .*

PROOF: Let M be a Riemannian manifold satisfying 5.1. with $\pi_1(M) = G$. We can always assume $\dim(M) \geq 3$. We will provide the argument under the weak technical condition that w is not a proper power. Let γ be a closed geodesic, representing w up to a conjugacy. Perturbing the metric, make γ to be embedded. Let $f : \Sigma^g \setminus D \rightarrow M$ be a map, realizing a given solution of $[x_1, y_1] \cdots [x_g, y_g] = w$ which exists by 1.6. We can find a conformal structure C on $\Sigma^g \setminus D$ and a harmonic map $\bar{f} : \Sigma^g \setminus D$, immersion near $\partial(\Sigma^g \setminus D)$ homotopic to f , by [15] and [16]. Moreover, it is easy to find a metric h in C , such that the boundary is geodesic (here we use that γ is a geodesic in M). So the conformal double Σ^{2g} is well-defined. Observe that there exists a conformal involution $\sigma : \Sigma^{2g} \rightarrow \Sigma^{2g}$ with $\partial D = \text{Fix}(\sigma)$. We claim that as the solution varies, the conformal structure of Σ^{2g} can be taken to stay in a compact set in M_{12g-6} . First observe that $\text{Area}(f) \leq \frac{2\pi(2g-1)}{k}$ by 6.2. Perturbing the metric of Σ as in [25] we may assume that $E^h(\bar{f}) = \text{Area}(f) \leq \frac{2\pi(2g-1)}{k}$. Adding a strip, we produce a map of Σ^{2g} to M , extending the map of $\Sigma^g \setminus D$ as shown in the Fig 6.6. An elementary computation shows that the energy of the extended map, say φ , stays bounded, when a is chosen big enough.

Let h be the unique hyperbolic metric in the conformal class, determined by this immersion. Observe that σ is necessarily an isometry of h . Consider now the sequence of solutions to $[x_1, y_1] \cdots [x_g, y_g] = w$ and let h_i be the corresponding hyperbolic metrics in Σ^{2g} and f_i, φ_i be the corresponding maps. If $[f_i]$ escapes all compact sets in M_{12g-6} , then, as in [30], we get a closed geodesic δ_i of (Σ^{2g}, h_i) whose length decreases to zero, and,

again by the collar argument, we can find a curve δ'_i in the collar around δ_i , such that δ'_i is homotopic to δ and the image $\bar{f}_i(\delta'_i)$ has the length decreasing to zero, and, in particular, is null-homotopic in M .

Consider two cases:

- (i) δ'_i does not intersect ∂D . Then we can cut Σ^g along δ'_i and make the genus reduction procedure as in 1. which contradicts $g = \text{genus}(w)$.
- (ii) δ'_i intersects ∂D . Let ε be a part of δ which lies in $\Sigma^g \setminus \partial D$, with ends in ∂D , say, p and q . Then we get two different segments $\bar{f}_i(\varepsilon)$ and a segment γ_0 of γ , joining $\bar{f}(p_i)$ and $\bar{f}(q_i)$, one of which, namely $\bar{f}_i(\varepsilon)$, has the length which decreases to zero and the other lies on the fixed closed geodesic. Hence $\text{length}(\gamma_0) \rightarrow 0$. If ε and γ_0 are not homotopic (with fixed ends) in $\Sigma^g \setminus \partial D$ we repeat the same game: cut $\Sigma^g \setminus \partial D$ along $\varepsilon_i \cup \gamma_0$, glue to $\bar{f}_i(\varepsilon_i \cup \gamma_0)$ a disk in M and reduce the genus of γ which is impossible. So any ε_i should be homotopic to a part of γ . The same applies to the mirror of $\Sigma^g \setminus D$ and hence δ'_i is homotopic to an multiple of ∂D which is impossible since $\bar{f}_i(\delta'_i)$ is null-homotopic in M . So $[f_i]$ stays within the compact set in M_{12g-6} . Twisting by a diffeomorphism of Σ^{2g} , we can assume that f_i stays in a compact set \mathcal{E} in T_{12g-6} . Now, there are only finitely many isotopic classes of conformal involutions for metrics in \mathcal{E} .

So there are only finite number of isotopy classes of simple curves, representing ∂D after the twist by a diffeomorphism of Σ^{2g} . Combining this with the finiteness theorem for $\varphi_{i*} : \pi_1(\Sigma^{2g}) \rightarrow G$, which follows from the argument of 5.3 complete the proof.

6.7. COROLLARY. *Let G be geometrically hyperbolic, let $w \in G'$ with $\text{genus}(w) = g$ and let A_w the stabilizer of w in $\text{Aut}(G)$. Then there exists a subgroup of A_w of finite index, which fixes a subgroup $H \triangleleft G$ of rank $\leq 2g$ such that $w \in H'$. In particular, if G is free and w is a simple commutator, $w \neq 1$, then there exists $N = N(w)$ which that for any endomorphism f of G , which fixes w , f^N fixes one (in fact any) rank two free group whose commutator contains w .*

PROOF: Any endomorphism $f : G \rightarrow G$ which fixes w , sends solutions of $[x_1, y_1] \cdots [x_g, y_g] = w$ to solutions. By 6.6. there are only finite number, say N , of subgroups, generated by

x_i, y_i , hence a result.

6.8. COROLLARY. *Let G be geometrically hyperbolic, and let $1 \neq u, v \in G'$. Then there does not exist an endomorphism φ of G such that $\varphi(u) = u$, $\varphi(v) = uv$.*

PROOF: For G free this is proved in [6], using 6.1.1. For any G , use 6.2. instead of 6.1.1. and complete the proof as in [6].

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Institute of Mathematics
The Hebrew University
Giv'at Ram, 91904 Jerusalem
ISRAEL

email:simplex@sunrise.huji.ac.il